

LIPSCHITZ PERTURBATIONS OF EXPANSIVE SYSTEMS

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ABSTRACT. We extend some known results from smooth dynamical systems to the category of Lipschitz homeomorphisms of compact metric spaces. We consider dynamical properties as robust expansiveness and structural stability allowing Lipschitz perturbations with respect to a hyperbolic metric. We also study the relationship between Lipschitz topologies and the C^1 topology on smooth manifolds.

1. INTRODUCTION

In the study of dynamical systems with discrete time we can distinguish two important categories: the *smooth* as for example C^1 diffeomorphisms of smooth manifolds and the *topological* as homeomorphisms of metric spaces. Here we will consider *Lipschitz* homeomorphisms, an intermediate one. Hyperbolicity is a key property of smooth systems. In this article we will consider Fathi's hyperbolic metric defined for an arbitrary expansive homeomorphism of a compact metric space to study its Lipschitz perturbations.

Recall that a homeomorphism $f: X \rightarrow X$ of a compact metric space is *expansive* if there is $\delta > 0$ such that $\text{dist}(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ implies $x = y$. In [2] Fathi proved that for every expansive homeomorphism there are a metric d , defining the original topology, and two parameters $\lambda > 1$ and $\varepsilon > 0$ such that if $d(x, y) < \varepsilon$ then $d(f(x), f(y)) \geq \lambda d(x, y)$ or $d(f^{-1}(x), f^{-1}(y)) \geq \lambda d(x, y)$. Fathi proposed to call such metric as an *adapted hyperbolic metric*. The hyperbolic metric is our starting point in the extension of results from hyperbolic diffeomorphisms to expansive homeomorphisms in the Lipschitz category.

A special type of diffeomorphisms with a hyperbolic behavior are the quasi-Anosov diffeomorphisms defined by Mañé. A diffeomorphism $f: M \rightarrow M$ of a compact smooth manifold is *quasi-Anosov* if its tangent map is expansive. Mañé in [10] proved that the interior in the C^1 topology of the set of expansive diffeomorphisms is the set of quasi-Anosov diffeomorphisms. In particular, every C^1 small perturbation of a quasi-Anosov diffeomorphism is expansive, i.e. they are C^1 *robustly expansive*. In Proposition 3.9 we will show that quasi-Anosov diffeomorphisms are robustly expansive even allowing Lipschitz perturbations.

Let us explain how we arrived to the Lipschitz category. The key is Theorem 6 in Walters paper [17]. There it is proved some kind of *structural stability* of transitive subshifts of finite type. We will show that Walters perturbations are in fact close in a Lipschitz topology, see Proposition 3.6. In particular he showed that these subshifts are robustly expansive. In this proof he uses the hyperbolicity of

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the usual distance for the shift map. Using similar ideas, in Corollary 3.8 we conclude that every expansive homeomorphism of a compact metric space is Lipschitz robustly expansive with respect to a hyperbolic metric. Another consequence of our results is that Anosov diffeomorphisms are structurally stable even allowing Lipschitz perturbations, see Corollary 4.7. That is, a homeomorphism obtained after a small Lipschitz perturbation is conjugate to the initial Anosov diffeomorphism. This result extends the one obtained in [16] as explained in Remark 4.8.

Structural stability is related with the shadowing or pseudo-orbit tracing property. See for example [17]. In [13] it is shown that a strong version of the shadowing property (called Lipschitz shadowing) is in fact equivalent with structural stability of C^1 diffeomorphisms. Expansive homeomorphisms of surfaces are known to be conjugate to pseudo-Anosov diffeomorphisms [9, 5] and if the surface is not a torus then pseudo-Anosov maps have not the shadowing property. But, as proved by Lewowicz in [8], they are persistent and, in some sense, C^1 structurally stable. See [8] for details and precise definitions. To prove the structural stability Lewowicz assumed that the perturbed homeomorphism coincides with the original one at the singular points of the stable foliation. See [8, Theorem 3.5]. We will show that pseudo-Anosov diffeomorphisms are Lipschitz-structurally stable with respect to a hyperbolic metric, see Corollary 4.9. In our setting we can also prove that singular points cannot be moved with a small Lipschitz perturbation, i.e., Lewowicz's hypothesis holds. This is done in Theorem 4.11.

Let us mention a very special property of the Lipschitz category that is not shared with diffeomorphisms. It is known that there are homeomorphisms of smooth manifolds that are not conjugate to smooth diffeomorphisms. An example can be obtained from the homeomorphism of \mathbb{R}^2 given in polar coordinates by

$$f(r, \theta) = (r, \theta + \sin(1/r)).$$

See [4] and references therein for more on this subject. In the C^2 category a well known example is the Denjoy C^1 circle diffeomorphism (with wandering points) that is not conjugate to a C^2 diffeomorphism. Consequently, from the point of view of topological dynamics, there is some loss of generality in assuming that the dynamic is generated by a C^r diffeomorphism. But, every homeomorphism, even if it is defined on a compact metric space, is conjugate with a bi-Lipschitz homeomorphism. Let us prove it. Given a homeomorphism $f: X \rightarrow X$ of a compact metric space (X, dist) , consider a new distance

$$d(x, y) = \sum_{n \in \mathbb{Z}} \frac{\text{dist}(f^n(x), f^n(y))}{2^{|n|}}.$$

Now it is easy to see that it defines the original topology and that f and f^{-1} are Lipschitz with respect to d . The conjugating homeomorphism is $id: (X, \text{dist}) \rightarrow (X, d)$. See [15] for more on this subject. Therefore, from the viewpoint of topological dynamics, there is no loss of generality in assuming that the homeomorphism is in fact bi-Lipschitz. We wish to present the following question: is every expansive homeomorphism of a compact smooth manifold conjugate to a C^1 diffeomorphism? This seems to be the case in all known examples.

This paper is organized as follows. In Section 2 we consider hyperbolic metrics for expansive homeomorphisms and we prove its robust expansiveness allowing perturbations in the topology considered by Walters in [17]. In Section 3 we introduce the Lipschitz topology in the space of bi-Lipschitz homeomorphisms and

we prove that it is equivalent with Walters topology. We conclude the Lipschitz robust expansiveness of every expansive homeomorphism with respect to a hyperbolic metric. In particular, quasi-Anosov diffeomorphisms are robustly expansive even allowing Lipschitz perturbations. In Section 4 we apply techniques from [17] to obtain the Lipschitz structural stability (with respect to a hyperbolic metric) of expansive homeomorphisms with the shadowing property. With similar techniques from [8] we obtain the same conclusion assuming that the homeomorphism is persistent instead of having the shadowing property. Applications to Anosov and pseudo-Anosov diffeomorphisms are given. For pseudo-Anosov homeomorphisms we consider a flat Riemannian metric with conical singularities as in, for example, [11, 6]. Using this metric we show that singular points of pseudo-Anosov homeomorphisms cannot be moved with a small Lipschitz perturbation. In Section 5 we consider smooth manifolds and we compare Lipschitz topologies with the usual C^1 topology.

2. HYPERBOLIC METRICS

Let $f: X \rightarrow X$ be a homeomorphism of a compact metrizable topological space.

Definition 2.1. A metric dist in X defining its topology is *f-hyperbolic* if there are $\delta > 0$ and $\lambda > 1$ such that if $\text{dist}(x, y) < \delta$ then

$$\max\{\text{dist}(f(x), f(y)), \text{dist}(f^{-1}(x), f^{-1}(y))\} \geq \lambda \text{dist}(x, y).$$

In this case δ is an *expansive constant* and λ is an *expanding factor* and we also say that f is *hyperbolic* with respect to dist .

Our first result, Theorem 2.7, is a generalization of [17, Theorem 6]. There it is proved some kind of robust expansiveness of the shift map. We will consider the topology introduced in [17]. Recall that a continuous function $f: X \rightarrow X$ is *Lipschitz* if there is $k > 0$ such that $\text{dist}(f(x), f(y)) \leq k \text{dist}(x, y)$ for all $x, y \in X$ and a homeomorphism f is *bi-Lipschitz* if f and f^{-1} are Lipschitz.

Remark 2.2. In [2, Theorem 5.1] Fathi proved that every expansive homeomorphism f defined on a compact metric space admits an *f-hyperbolic* metric defining the original topology of the space. Moreover, f is a bi-Lipschitz homeomorphism with respect to this hyperbolic metric.

Definition 2.3. Given two homeomorphisms f, g of X define the C^0 -metric as

$$(1) \quad \text{dist}_{C^0}(f, g) = \max_{x \in X} \text{dist}(f(x), g(x)) + \max_{x \in X} \text{dist}(f^{-1}(x), g^{-1}(x)).$$

Given two Lipschitz homeomorphisms $f, g: X \rightarrow X$ of a compact metric space (X, dist) consider the following pseudometric

$$\text{dist}'_W(f, g) = \sup_{x \neq y} \frac{|\text{dist}(f(x), f(y)) - \text{dist}(g(x), g(y))|}{\text{dist}(x, y)}.$$

Remark 2.4. It is not a metric. In fact it holds that $\text{dist}'_W(f, g) = 0$ if and only if $f \circ g^{-1}$ is an isometry of (X, dist) . This is a direct consequence of the definition.

If f, g are bi-Lipschitz homeomorphisms define the metric

$$(2) \quad \text{dist}_W(f, g) = \text{dist}_{C^0}(f, g) + \text{dist}'_W(f, g) + \text{dist}'_W(f^{-1}, g^{-1}).$$

Remark 2.5. Notice that dist_W depends on the metric dist of X .

Definition 2.6. A bi-Lipschitz homeomorphism $f: X \rightarrow X$ is *dist_W-robustly expansive* if there are $\varepsilon, \delta > 0$ such that if $\text{dist}_W(f, g) < \varepsilon$ then g is expansive with expansive constant δ .

Theorem 2.7. *Every hyperbolic homeomorphism is dist_W-robustly expansive.*

Proof. Let $f: X \rightarrow X$ be a hyperbolic homeomorphism with respect to the metric dist on X . Let $\delta > 0$ and $\lambda > 1$ be an expansive constant and an expanding factor respectively. Take $\varepsilon > 0$ and $\lambda' > 1$ such that $\lambda - \varepsilon > \lambda'$. We will show that λ' is an expanding factor and δ is an expansive constant for g if $\text{dist}_W(f, g) < \varepsilon$. Take $x, y \in X$ such that $\text{dist}(x, y) \in (0, \delta)$ and assume that $\text{dist}(f(x), f(y)) \geq \lambda \text{dist}(x, y)$ (the case $\text{dist}(f^{-1}(x), f^{-1}(y)) \geq \lambda \text{dist}(x, y)$ is similar). Since $\text{dist}_W(f, g) < \varepsilon$, we know that

$$\left| \frac{\text{dist}(f(x), f(y))}{\text{dist}(x, y)} - \frac{\text{dist}(g(x), g(y))}{\text{dist}(x, y)} \right| < \varepsilon < \lambda - \lambda'.$$

Then

$$\lambda' \leq \lambda' + \frac{\text{dist}(f(x), f(y))}{\text{dist}(x, y)} - \lambda < \frac{\text{dist}(g(x), g(y))}{\text{dist}(x, y)}$$

because

$$0 \leq \frac{\text{dist}(f(x), f(y))}{\text{dist}(x, y)} - \lambda.$$

Then $\lambda' \text{dist}(x, y) \leq \text{dist}(g(x), g(y))$. Therefore δ is an expansive constant and λ' is an expanding factor for g . \square

Corollary 2.8. *For every expansive homeomorphism f of a compact metrizable space X there is a metric dist in X defining its topology and making f a dist_W-robustly expansive bi-Lipschitz homeomorphism.*

Proof. It follows by [2, Theorem 5.1] (see Remark 2.2) and Theorem 2.7. \square

3. LIPSCHITZ TOPOLOGY

The purpose of this section is to prove that dist_W induces a Lipschitz topology in the space of bi-Lipschitz homeomorphisms of X .

Definition 3.1. Given two bi-Lipschitz homeomorphisms $f, g: X \rightarrow X$ of a compact metric space (X, dist) we define

$$(3) \quad \text{dist}'_L(f, g) = \sup_{x \neq y} \left| \log \left(\frac{\text{dist}(f(x), f(y))}{\text{dist}(g(x), g(y))} \right) \right|$$

and the *Lipschitz metric* as

$$\text{dist}_L(f, g) = \text{dist}_{C^0}(f, g) + \text{dist}'_L(f, g) + \text{dist}'_L(f^{-1}, g^{-1}).$$

Remark 3.2. As for dist'_W , it holds that $\text{dist}'_L(f, g) = 0$ if and only if $f \circ g^{-1}$ is an isometry.

Let us explain why we call it Lipschitz metric.

Definition 3.3. A *normed group* is a group G with identity $e \in G$ and a function $\|\cdot\|: G \rightarrow \mathbb{R}$ satisfying:

- (1) $\|f\| \geq 0$ for all $f \in G$ with equality only at $f = e$,
- (2) $\|f\| = \|f^{-1}\|$ for all $f \in G$ and
- (3) $\|fg\| \leq \|f\| + \|g\|$ for all $f, g \in G$.

Every norm induces a distance as

$$\text{dist}_{\|\cdot\|}(f, g) = \|fg^{-1}\| + \|g^{-1}f\|.$$

Remark 3.4. In some sense, we measure how far are fg^{-1} and $g^{-1}f$ from the identity. In [18] it is called as an *operational metric*.

For example, in the group of homeomorphisms of X we can consider the C^0 -norm

$$\|f\|_{C^0} = \max_{x \in X} \text{dist}(x, f(x)).$$

And we obtain that dist_{C^0} of Equation (1) is the distance induced by the C^0 -norm. See for example [1] for more on normed groups.

Let $\mathcal{L} = \mathcal{L}(X, \text{dist})$ denote the group of bi-Lipschitz homeomorphisms of a compact metric space (X, dist) . The operation in \mathcal{L} is composition. As usual, define the *Lipschitz constant* of $f \in \mathcal{L}$ as

$$\text{lips}(f) = \sup_{x \neq y} \frac{\text{dist}(f(x), f(y))}{\text{dist}(x, y)}.$$

It is easy to see that:

- (1) $\text{lips}(f) \geq 1$ for all $f \in \mathcal{L}$ and
- (2) $\text{lips}(f \circ g) \leq \text{lips}(f) \text{lips}(g)$ for all $f, g \in \mathcal{L}$.

In light of these properties, it is natural to define $\text{loglips}(f) = \log(\text{lips}(f))$ and to consider

$$\|f\|'_L = \max\{\text{loglips}(f), \text{loglips}(f^{-1})\}.$$

We have that $(\mathcal{L}, \|\cdot\|'_L)$ is a semi-normed group. In fact $\|f\|'_L = 0$ if and only if f is an isometry. In order to obtain a norm we define

$$\|f\|_L = \|f\|_{C^0} + \|f\|'_L.$$

The following proposition explains why we say that dist_L from Equation (3) is a Lipschitz metric in \mathcal{L} .

Proposition 3.5. *It holds that $\text{dist}_{\|\cdot\|_L} = \text{dist}_L$.*

Proof. By the definitions we have that

$$\begin{aligned} \text{dist}_{\|\cdot\|_L}(f, g) &= \|fg^{-1}\|_L + \|g^{-1}f\|_L \\ &= \|fg^{-1}\|_{C^0} + \|fg^{-1}\|'_L + \|g^{-1}f\|_{C^0} + \|g^{-1}f\|'_L \\ &= \text{dist}_{C^0}(f, g) + \|fg^{-1}\|'_L + \|g^{-1}f\|'_L \end{aligned}$$

and

$$\text{dist}_L(f, g) = \text{dist}_{C^0}(f, g) + \text{dist}'_L(f, g) + \text{dist}'_L(f^{-1}, g^{-1}).$$

Therefore, it is sufficient to prove that $\text{dist}'_L(f, g) = \|fg^{-1}\|'_L$. A similar argument will give us $\text{dist}'_L(f^{-1}, g^{-1}) = \|g^{-1}f\|'_L$. Applying the definitions we have:

$$\begin{aligned} \|fg^{-1}\|'_L &= \max\{\text{loglips}(fg^{-1}), \text{loglips}(gf^{-1})\} \\ &= \max\left\{\sup_{x \neq y} \log \frac{\text{dist}(fg^{-1}(x), fg^{-1}(y))}{\text{dist}(x, y)}, \sup_{x \neq y} \log \frac{\text{dist}(gf^{-1}(x), gf^{-1}(y))}{\text{dist}(x, y)}\right\} \\ &= \max\left\{\sup_{x \neq y} \log \frac{\text{dist}(f(x), f(y))}{\text{dist}(g(x), g(y))}, \sup_{x \neq y} \log \frac{\text{dist}(g(x), g(y))}{\text{dist}(f(x), f(y))}\right\} \\ &= \sup_{x \neq y} \left| \log \left(\frac{\text{dist}(f(x), f(y))}{\text{dist}(g(x), g(y))} \right) \right| = \text{dist}'_L(f, g). \end{aligned}$$

And the proof ends. \square

Proposition 3.6. *The metrics dist_L and dist_W define the same topology on \mathcal{L} .*

Proof. For $f \in \mathcal{L}$ define

$$\delta_0 = \min\{1/\text{lips}(f), 1/\text{lips}(f^{-1})\}$$

and

$$\delta_1 = \max\{\text{lips}(f), \text{lips}(f^{-1})\}.$$

Given $\delta \in (0, \delta_0)$ consider $k > 0$ such that if $u, v \in [\delta_0 - \delta, \delta_0 + \delta]$ then

$$|\log u - \log v| \leq k|u - v|.$$

Then, with standard techniques, it can be proved that if $\text{dist}_W(f, g) < \delta$ then $\text{dist}_L(f, g) < k\delta$. For this purpose, one has to note that

$$\log \left(\frac{\text{dist}(f(x), f(y))}{\text{dist}(g(x), g(y))} \right) = \log \left(\frac{\text{dist}(f(x), f(y))}{\text{dist}(x, y)} \right) - \log \left(\frac{\text{dist}(g(x), g(y))}{\text{dist}(x, y)} \right).$$

The other part of the proof follows by similar arguments. \square

Definition 3.7. A bi-Lipschitz homeomorphism $f \in \mathcal{L}$ is *Lipschitz-robustly expansive* if there are $\varepsilon, \delta > 0$ such that if $\text{dist}_L(f, g) < \varepsilon$ then g is expansive with expansive constant δ .

Corollary 3.8. *Every hyperbolic homeomorphism is Lipschitz-robustly expansive. Or equivalently, every expansive homeomorphism of a compact metric space is Lipschitz robustly expansive with respect to an f -hyperbolic metric.*

Proof. It is a direct consequence of Theorem 2.7 and Proposition 3.6. \square

Recall that a C^1 diffeomorphism $f: M \rightarrow M$ is *quasi-Anosov* if the set $\{\|df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded for all $v \neq 0$. It is known [10] that quasi-Anosov diffeomorphisms are C^1 -robustly expansive (considering C^1 perturbations).

Proposition 3.9. *Quasi-Anosov diffeomorphisms are Lipschitz-robustly expansive with respect to every Riemannian metric dist_g .*

Proof. Let f be a quasi-Anosov diffeomorphism. By Corollary 3.8 it would be sufficient to prove that dist_g is hyperbolic for f , but this may not be true because the Riemannian metric may not be an adapted metric. Instead, we will show that dist_g is hyperbolic for f^n for some $n \geq 1$. This is sufficient too. Consider n such that for all $v \neq 0$, $v \in TM$, it holds that $\|df^n(v)\| > 2\|v\|$ or $\|df^{-n}(v)\| > 2\|v\|$. Such n exists, as it is proved in [7, Lemma 2.3] (there it is assumed that f is Anosov but it only uses that f is quasi-Anosov). Now, using the exponential map of the Riemannian metric, we can locally lift f to the tangent fiber bundle. In this way we can view f as a small Lipschitz perturbation of df . Consequently, dist_g is hyperbolic for f^n . \square

4. LIPSCHITZ STRUCTURAL STABILITY

In this section we will consider some shadowing properties to study the structural stability allowing Lipschitz perturbations.

Definition 4.1. We say that $f \in \mathcal{L}(X, \text{dist})$ is *Lipschitz structurally stable* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\text{dist}_L(f, g) < \delta$ then there is a homeomorphism $h: X \rightarrow X$ such that $fh = hg$ and $\text{dist}_{C^0}(h, \text{id}) < \varepsilon$.

Definition 4.2. A homeomorphism $f: X \rightarrow X$ has the *weak shadowing property* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\text{dist}_{C^0}(f, g) < \delta$ then for all $x \in X$ there is $y \in X$ such that $\text{dist}(f^n(y), g^n(x)) < \varepsilon$ for all $n \in \mathbb{Z}$.

In [12] the expression *weak shadowing property* is considered with a different meaning. Let us recall that a δ -pseudo-orbit is a sequence $x_n \in X$ with $n \in \mathbb{Z}$ such that $\text{dist}(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$. A homeomorphism $f: X \rightarrow X$ has the (usual) *shadowing property* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\{x_n\}_{n \in \mathbb{Z}}$ is a δ -pseudo-orbit then there is $y \in X$ such that $\text{dist}(f^n(y), x_n) < \varepsilon$ for all $n \in \mathbb{Z}$. This is also called as *pseudo-orbit tracing property*.

Remark 4.3. The shadowing property implies the weak shadowing property. Theorems 4, 5 and 6 in [17] are stated with the shadowing property but the proofs only need the weak shadowing property. To see that this is true, the reader should check that the proofs of Theorems 5 and 6 uses the shadowing property via Theorem 4, and in the proof of Theorem 4 it is used, in its first paragraph, considering a pseudo-orbit that is a true orbit of the perturbed homeomorphism there called as S .

In the general setting of metric spaces, pseudo-orbits may be far from being real orbits of C^0 perturbations as we explain with the following example.

Remark 4.4. On compact metric spaces there are homeomorphisms with the weak shadowing property but without the usual shadowing property. Let us give an example. Let $f: X \rightarrow X$ be a homeomorphism of a compact metric space X with three fixed points $a, b, c \in X$. Also assume that there are $I \subset X$ and $y \in X$ such that I is a compact arc, I is also an open subset of X , $\lim_{n \rightarrow -\infty} f^n(I) = a$, $\lim_{n \rightarrow \infty} f^n(I) = b = \lim_{n \rightarrow -\infty} f^n(y)$, $\lim_{n \rightarrow \infty} f^n(y) = c$ and

$$X = \{a, b, c\} \cup \{f^n(y) : n \in \mathbb{Z}\} \cup \bigcup_{n \in \mathbb{Z}} f^n(I).$$

See Figure 1. It is easy to see that f has not the usual shadowing property because

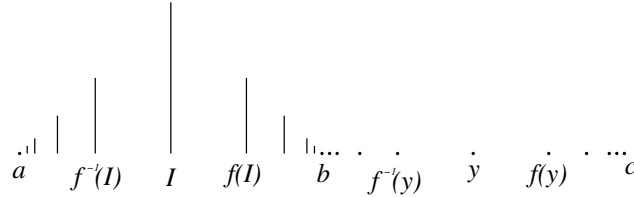


FIGURE 1. The space X is a countable union of points and intervals. The homeomorphism f moves isolated points and intervals to the right, while the accumulation points a, b, c are fixed.

there are pseudo-orbits starting at a and finishing at c without real orbits tracing it. In order to prove that f has the weak shadowing property we have to note that every homeomorphism of X , in particular a perturbation of f , has to fix a, b, c and preserve the left part of b in the figure.

Let us give another shadowing property that was introduced in [8].

Definition 4.5. A homeomorphism $f: X \rightarrow X$ is *persistent* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\text{dist}_{C^0}(f, g) < \delta$ then for all $x \in X$ there is $y \in X$ such that $\text{dist}(f^n(x), g^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$.

Note the difference between this definition and the weak shadowing property. It is known that pseudo-Anosov homeomorphisms of surfaces are persistent but they have not the weak shadowing property.

The proof of the following result uses well known techniques that can be found in [17] for the case of the shadowing property. For the case of f persistent see [8, Section 1] (Lemma 1.2 and the Remark below it).

Theorem 4.6. *Let $f \in \mathcal{L}(M, \text{dist})$ with dist an f -hyperbolic metric and M a compact manifold without boundary. If f is persistent or has the weak shadowing property then f is Lipschitz structurally stable.*

Proof. Since dist is f -hyperbolic we know by Corollary 3.8 that f is Lipschitz robustly expansive. Therefore, there are $\sigma, \alpha > 0$ such that if $\text{dist}_L(f, g) < \sigma$ then g is expansive with expansive constant α . In order to prove that f is Lipschitz structurally stable take $\varepsilon \in (0, \alpha/2)$. Since M is a compact manifold without boundary, we can assume that every continuous function $h: M \rightarrow M$ such that $\text{dist}_{C^0}(h, \text{id}_M) < \varepsilon$ is surjective. Assume that f has the weak shadowing property. Therefore, there is $\delta \in (0, \sigma)$ such that if $\text{dist}_{C^0}(f, g) < \delta$ then for all $x \in M$ there is $h(x) \in M$ such that

$$(4) \quad \text{dist}(f^n(h(x)), g^n(x)) < \varepsilon \text{ for all } n \in \mathbb{Z}.$$

In this way we have a function $h: M \rightarrow M$. Assume that $\text{dist}_L(f, g) < \delta$. If $h(x) = h(x')$ then $\text{dist}(f^n(h(x)), g^n(x)) < \varepsilon$ and $\text{dist}(f^n(h(x')), g^n(x')) < \varepsilon$ for all $n \in \mathbb{Z}$. Therefore, $\text{dist}(g^n(x), g^n(x')) < \alpha$ for all $n \in \mathbb{Z}$. Since α is an expansive constant for g we have that $x = x'$. Consequently, h is injective. Let us prove that h is continuous. Assume that $x_n \rightarrow x$ and $h(x_n) \rightarrow y$. Taking limit, we have that $\text{dist}(f^n(y), g^n(x)) \leq \varepsilon$ for all $n \in \mathbb{Z}$. Also, we know that $\text{dist}(f^n(h(x)), g^n(x)) < \varepsilon$. Then $\text{dist}(f^n(h(x)), f^n(y)) < \alpha$ for all $n \in \mathbb{Z}$. Since α is an expansive constant for f we conclude that $y = h(x)$. Therefore h is continuous. Notice that by (4) we have that $\text{dist}_{C^0}(h, \text{id}_M) < \varepsilon$ ($n = 0$). Therefore, h is surjective and a homeomorphism. In order to show that $f \circ h = h \circ g$ fix $x \in M$ and note that by (4) it holds that

$$\text{dist}(f^{n+1}(h(x)), g^{n+1}(x)) = \text{dist}(f^n(f(h(x))), g^n(g(x))) < \varepsilon$$

for all $n \in \mathbb{Z}$. Applying (4) again, we have that

$$\text{dist}(f^n(h(g(x))), g^n(g(x))) < \varepsilon$$

for all $n \in \mathbb{Z}$. And by the triangular inequality

$$\text{dist}(f^n(f(h(x))), f^n(h(g(x)))) < \alpha$$

for all $n \in \mathbb{Z}$. Then $f(h(x)) = g(h(x))$ for all $x \in M$ because α is an expansive constant of f .

The case of f persistent is similar, using the condition: $\text{dist}(f^n(x), g^n(h(x))) < \varepsilon$ for all $n \in \mathbb{Z}$, instead of (4). \square

Notice that in the previous proof we only used that M is a compact manifold, instead of an arbitrary compact metric space, to ensure that small perturbations of the identity are onto.

Corollary 4.7. *Anosov diffeomorphisms are Lipschitz structurally stable with respect to any Riemannian metric.*

Proof. It is well known that Anosov diffeomorphisms have the shadowing property. If the Riemannian metric is not adapted for f , we consider an adapted one. Note that all Riemannian metrics are Lipschitz equivalent. In this way the Lipschitz topology on bi-Lipschitz homeomorphisms of M does not change and now the Riemannian metric dist_g is hyperbolic for f . Then, by Theorem 4.6 the corollary follows. \square

Remark 4.8. In [16] a result similar to Corollary 4.7 was obtained using a Lipschitz topology defined via local charts. As we will see in Section 5 our Lipschitz topology (induced by dist_W) allows more perturbations. Consequently, Corollary 4.7 is stronger than the result in [16].

Corollary 4.9. *Pseudo-Anosov maps of compact surfaces are Lipschitz structurally stable with respect to a hyperbolic metric.*

Proof. In [8] it is shown that Pseudo-Anosov maps are persistent. Therefore we conclude by Theorem 4.6. \square

Remark 4.10. For a pseudo-Anosov map f of a compact surface S denote by $\text{Sing}(f)$ the set of singular points of the stable foliation of f . Consider a flat metric in $S \setminus \text{Sing}(f)$ such that stable and unstable leaves are geodesics. This Riemannian metric induces a f -hyperbolic distance in S . The construction is standard and the details can be found for example in [6, Section 3.1]. An important property of this metric, for our purpose, is that the length of the boundary of a small ball of radius r centered at a singular point p equals $n\pi r$, where n is the number of stable prongs of p .

In [8, Theorem 3.5] Lewowicz obtained a result similar to Corollary 4.9 but in the C^1 category. In his hypothesis it is required that the perturbed map coincides with the original one at singular points. In the Lipschitz setting, with a hyperbolic metric, this hypothesis is not needed as the next result shows.

Theorem 4.11. *Assume that $f: S \rightarrow S$ is a pseudo-Anosov homeomorphism of a compact surface with a hyperbolic metric as in Remark 4.10. Then there is $\varepsilon > 0$ such that if $\text{dist}_L(f, g) < \varepsilon$ then $\text{Sing}(g) = \text{Sing}(f)$.*

Proof. If f and g are Lipschitz close then $j = f \circ g^{-1}$ and j^{-1} are Lipschitz close to the identity of S . Then, we can take $\varepsilon > 0$ such that if $\text{dist}_L(f, g) < \varepsilon$ then

$$(5) \quad \text{lips}(j) \text{lips}(j^{-1}) < 3/2.$$

Let us first show that if $\text{dist}_L(f, g) < \varepsilon$ then $j(\text{Sing}(f)) = \text{Sing}(f)$. Assume by contradiction that there is $x \notin \text{Sing}(f)$ but $j(x) \in \text{Sing}(f)$ with $\text{dist}_L(f, g) < \varepsilon$. Consider C_1 a small circle of radius r_1 and centered at x such that $C_1 \cap j(C_1) = \emptyset$. Let $r_2 = \min_{y \in j(C_1)} \text{dist}(j(x), y)$ and denote by C_2 the circle centered at $j(x)$ with radius r_2 as shown in Figure 2. Since x is a regular point (of the metric) and the metric is flat we have that $\text{len}(C_1) = 2\pi r_1$, where len denotes the length of the curve. Since $j(x)$ is a singular point we have that

$$(6) \quad \text{len}(C_2) \geq 3\pi r_2$$

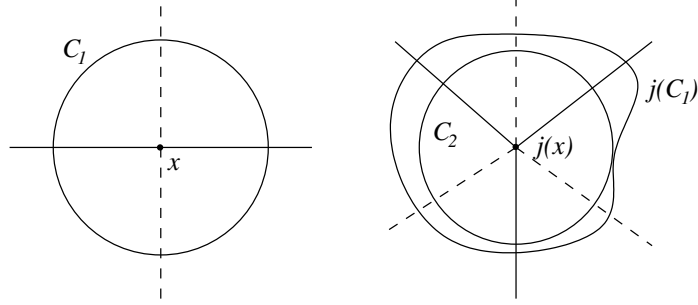


FIGURE 2. A small Lipschitz perturbation cannot move a singular point.

because a neighborhood of $j(x)$ is constructed by gluing at least 3 flat half-planes as explained in Remark 4.10. We also have that

$$(7) \quad \text{len}(j(C_1)) \leq \text{lips}(j) \text{len}(C_1) = \text{lips}(j) 2\pi r_1.$$

By the isoperimetric inequality we know that $\text{len}(j(C_1)) \geq \text{len}(C_2)$. This and (6) implies that

$$(8) \quad \text{len}(j(C_1)) \geq 3\pi r_2.$$

Finally, by (7) and (8) we have $3r_2 \leq \text{lips}(j) 2r_1$. Now take $z \in C_1$ such that $r_2 = \text{dist}(j(x), j(z))$. Since $z \in C_1$ we have that $\text{dist}(x, z) = r_1$. Then

$$\text{lips}(j^{-1}) \geq \frac{\text{dist}(x, z)}{\text{dist}(j(x), j(z))} = r_1/r_2.$$

But, since $r_1/r_2 \geq \frac{3}{2\text{lips}(j)}$ we have that $\text{lips}(j) \text{lips}(j^{-1}) \geq 3/2$. This contradicts inequality (5) and proves that $j(\text{Sing}(f)) = \text{Sing}(f)$.

Since $j = fg^{-1}$ and $\text{Sing}(f)$ is f -invariant we have that $g(\text{Sing}(f)) = \text{Sing}(f)$. To conclude the proof we have to note that $\text{Sing}(f)$ and $\text{Sing}(g)$ are g -invariant finite sets and they are close if ε is small. Therefore, since g is expansive, we conclude that $\text{Sing}(f) = \text{Sing}(g)$ as we wanted to prove. \square

5. TOPOLOGIES ON THE SPACE OF SMOOTH FUNCTIONS

In [17, page 238] Walters considered dist_W as “analogous to a C^1 metric”. In this section we will investigate the relationship between the C^1 topology and two different Lipschitz topologies. In \mathbb{R}^n we know that every C^1 diffeomorphism is Lipschitz, therefore, we can restrict a Lipschitz topology to the space of C^1 diffeomorphisms. A natural question is: if two diffeomorphisms are close in a Lipschitz metric are they necessarily C^1 close? This is the problem that will be considered in this section for dist_W , defined at (2) above, and the *natural* Lipschitz metric induced by a vector space structure that will be defined below. In general, the convergence in the C^1 topology implies the convergence in the Lipschitz topology, if we consider smooth manifolds with Riemannian metrics.

For the first result denote by $I = [0, 1]$ the unit interval with the usual metric $\text{dist}(x, y) = |y - x|$ for all $x, y \in I$. Denote by $\text{Diff}^1(I)$ the group of C^1 diffeomorphisms of I . The C^1 topology in the one-dimensional case is defined by the distance

$\text{dist}_{C^0}(f, g) + \text{dist}_{C^0}(f', g')$ for $f, g \in \text{Diff}^1(I)$. We refer the reader to [14] for the definition of the C^1 topology on general manifolds.

Proposition 5.1. *In $\text{Diff}^1(I)$ the metric dist_W defines the C^1 topology.*

Proof. If $\text{dist}_{C^0}(f, g) < \varepsilon < 1/2$ then both diffeomorphisms have derivatives of the same sign. Suppose that f and g are increasing, the other case being similar. Assume that $\text{dist}_W(f, g) \leq \varepsilon$. Therefore

$$\sup_{x < y} \frac{|f(y) - f(x) - g(y) + g(x)|}{y - x} \leq \varepsilon.$$

Fixing x and taking limit as $y \rightarrow x^+$ we have that $|f'(x) - g'(x)| \leq \varepsilon$. Since this holds for all $x \in I$ we have that f and g are C^1 -close. \square

Remark 5.2. The argument of the previous proof applied for diffeomorphisms of a compact ball in \mathbb{R}^n will give us $||d_p f(v) - d_p g(v)|| \leq \varepsilon$. But this does not imply that $||d_p f(v) - d_p g(v)|| \leq \varepsilon$.

Let us show that the previous proposition does not hold for higher dimensional manifolds. For $\varepsilon > 0$ given we will construct a diffeomorphism $f: D \rightarrow D$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, so that $\text{dist}'_W(f, \text{id}) < \varepsilon$ and $d_{(0,0)}f$ is a rotation of angle π . For this purpose we need the following preliminary result.

Lemma 5.3. *For all $\varepsilon > 0$ there are a C^∞ function $\gamma: [0, 1] \rightarrow [0, \pi]$ and $0 < a < b < \varepsilon$ such that:*

- (1) $\gamma(r) = \pi$ for all $r \in [0, a]$,
- (2) $\gamma(r) = 0$ for all $r \in [b, 1]$,
- (3) $0 \leq |r\gamma'(r)| \leq \varepsilon$ for all $r \geq 0$.

Proof. For $k \in \mathbb{R}$ consider the function $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ given by $g(r) = k - \varepsilon \log(r)$. Notice that g satisfies the differential equation $rg'(r) = -\varepsilon$ for all $r > 0$. It is easy to see that if $k < 0$ and $|k|$ is sufficiently large then there is $d \in (0, \varepsilon)$ such that $g(d) = 0$. Fix this value of k and take $c \in (0, d)$ so that $g(c) = \pi$. Now using bump

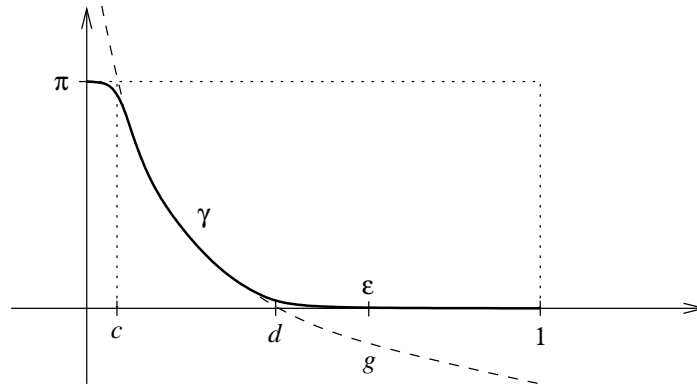


FIGURE 3. The function γ of Lemma 5.3. In dotted line the function g .

functions near the points c and d it is easy to obtain the function γ . In Figure 3 the construction is illustrated. The numbers a and b are obtained in $(0, c)$ and (d, ε) respectively. \square

Consider $\varepsilon > 0$ given and the function γ given by Lemma 5.3. Define in polar coordinates the diffeomorphism $f: D \rightarrow D$ as

$$f(r \cos \theta, r \sin \theta) = r(\cos(\theta + \gamma(r)), \sin(\theta + \gamma(r))).$$

Denote by $R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the rotation of angle θ around the origin. Notice that $f(p) = R_\pi(p)$ if $\|p\| \leq a$ and $f(p) = p$ if $\|p\| \geq b$. Also, $\|f(p)\| = \|p\|$ for all $p \in D$. Therefore, f is C^0 close to the identity of D if ε is small and for all ε the differential $d_{(0,0)}f = R_\pi$. Consequently f is C^1 far from the identity. We will show that f is close to the identity for the metric dist_W . For this, we will first show that the differential of f is close to a rotation at each $p \in D$.

Lemma 5.4. *For all $p \in D$ it holds that $\|d_p f - R_{\gamma(r)}\| \leq \varepsilon$ where $r = \|p\|$.*

Proof. For $\|p\| < a$ or $\|p\| > b$ we know that $d_p f = R_{\gamma(r)}$. By the symmetry of the construction, it is no loss of generality to consider $p = (r, 0)$ with $r > 0$. In the standard basis of \mathbb{R}^2 the differential of f is given by the matrix

$$d_p f = \begin{pmatrix} \cos(\gamma(r)) - r\gamma'(r) \sin(\gamma(r)) & -\sin(\gamma(r)) \\ \sin(\gamma(r)) + r\gamma'(r) \cos(\gamma(r)) & \cos(\gamma(r)) \end{pmatrix}$$

and

$$d_p f - R_{\gamma(r)} = \begin{pmatrix} -r\gamma'(r) \sin(\gamma(r)) & 0 \\ r\gamma'(r) \cos(\gamma(r)) & 0 \end{pmatrix}.$$

By construction we have that $|r\gamma'(r)| \leq \varepsilon$ and then $\|d_p f - R_{\gamma(r)}\| \leq \varepsilon$. \square

Proposition 5.5. *For the diffeomorphism f it holds that $\text{dist}'_W(f, \text{id}_D) \leq \varepsilon$.*

Proof. Given $p \neq q$ in D consider the line $\alpha(t) = p + t(q - p)$ for $0 \leq t \leq 1$. Then

$$\begin{aligned} \|f(q) - f(p)\| &\leq \int_0^1 \left\| \frac{d}{dt} f(\alpha(t)) \right\| dt \\ &\leq \int_0^1 (1 + \varepsilon) \left\| \frac{d}{dt} \alpha(t) \right\| dt \\ &\leq (1 + \varepsilon) \|q - p\|. \end{aligned}$$

Notice that

$$f^{-1}(r \cos \theta, r \sin \theta) = r(\cos(\theta - \gamma(r)), \sin(\theta - \gamma(r))).$$

Therefore similar estimates gives us

$$\|q - p\| \leq (1 + \varepsilon) \|f(q) - f(p)\|$$

and

$$\frac{1}{1 + \varepsilon} \leq \frac{\|f(q) - f(p)\|}{\|q - p\|} \leq 1 + \varepsilon$$

This implies that $\text{dist}'_W(f, \text{id}_D) \leq \varepsilon$. \square

Therefore we have proved:

Theorem 5.6. *There is a sequence of C^1 diffeomorphisms f_n of the compact disk $D \subset \mathbb{R}^2$ do not converging to the identity in the C^1 topology but satisfying that $\text{dist}_W(f_n, \text{id}_D) \rightarrow 0$ as $n \rightarrow \infty$.*

Let us now consider another Lipschitz topology. Suppose that $M \subset \mathbb{R}^n$ is a compact C^1 manifold without boundary. Denote by $\|\cdot\|$ the Euclidean norm in \mathbb{R}^n . For $f: M \rightarrow M$ a bi-Lipschitz homeomorphisms define

$$\text{lips}(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

Theorem 5.7. *The distance*

$$\text{dist}_{C^1}(f, g) = \text{dist}_{C^0}(f, g) + \text{lips}(f - g)$$

defines the C^1 topology of C^1 diffeomorphisms of $M \subset \mathbb{R}^n$.

Proof. Suppose that $g, f_n: M \rightarrow M$ are C^1 diffeomorphisms, $n \in \mathbb{N}$, such that $\lim_{n \rightarrow \infty} \text{dist}_{C^1}(f_n, g) = 0$. Taking local charts we can assume that $g, f_n: B_r \subset \mathbb{R}^k \rightarrow \mathbb{R}^k$ where B_r is a compact ball and $k = \dim(M)$. Now it is easy to see that

$$\|d_p f_n - d_p g\| \leq \text{lips}(f_n - g)$$

for all $p \in B_r$. Therefore $d_p f_n$ converges to $d_p g$ uniformly in $p \in B_r$. \square

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